Published by Institute of Physics Publishing for SISSA/ISAS
Received: *August 23, 2004*
Revised: *October 12, 2004*
Accepted: *October 14, 2004*

Super-Pohlmeyer invariants and boundary states for non-abelian gauge fields

Urs Schreiber
Universität Duisburg-Essen
Essen, 45117, Germany
E-mail: Urs.Schreiber@uni-essen.de

Abstract: Aspects of the supersymmetric extension of the Pohlmeyer invariants are studied, and their relation to superstring boundary states for non-abelian gauge fields is discussed.
We show that acting with a super-Pohlmeyer invariant with respect to some non-abelian gauge field A (which has to be constant due to the definition of the Pohlmeyer invariants) on the boundary state of a bare D9 brane produces the boundary state describing that non-abelian background gauge field on the brane if the only non-trivial commutators among the components of A are those involving a single lightlike component. Known consistency conditions on that boundary state equivalent to the background equations of motion for A hence also apply to the quantized Pohlmeyer invariants.

Keywords: Superstrings and Heterotic Strings, D-branes, Integrable Field Theories.

 http://jhep.sissa.it/archive/papers/jhep102004035/jhep102004035.pdf

Contents

1. Introduction

This paper demonstrates a relation between two apparently unrelated aspects of superstrings: boundary states for nonabelian gauge fields and (super-)Pohlmeyer invariants.

On the one hand side superstring boundary states describing excitations of non-abelian gauge fields on D-branes are still the subject of investigations [1, 2, 3] and are of general interest for superstring theory, as they directly mediate between string theory and gauge theory.

On the other hand, studies of string quantization focusing on non-standard worldsheet invariants, the so-called Pohlmeyer invariants, done in [4]–[7] and recalled in [8], were shown in [9, 10] to be related to the standard quantization of the string by way of the well-known DDF invariants. This raised the question whether the Pohlmeyer invariants are of any genuine interest in (super-)string theory as commonly understood.

Here it shall be shown that the (super-)Pohlmeyer invariants do indeed play an interesting role as boundary state deformation operators for non-abelian gauge fields, thus connecting the above two topics and illuminating aspects of both them.

A boundary state is a state in the closed string's Hilbert space constructed in such a way that inserting the vertex operator of that state in the path integral over the sphere reproduces the disk amplitudes for certain boundary conditions (D-branes) of the open string. In accord with the general fact that the worldsheet path integral insertions which describe background field excitations are exponentiations of the corresponding vertex operators, it turns out that the boundary states which describe gauge field excitations on the D-brane have the form of (generalized) Wilson lines of the gauge field along the closed string [11, 12, 1, 2, 3].

JHEP10(2004)035

Long before these investigations, it was noted by Pohlmeyer [7], in the context of the classical string, that generalized Wilson lines along the closed string with respect to an auxiliary gauge connection on spacetime provide a "complete" set of invariants of the theory, i.e. a complete set of observables which (Poisson-)commute with all the Virasoro constraints.

Given these two developments it is natural to suspect that there is a relation between Pohlmeyer invariants and boundary states. Just like the DDF invariants (introduced in [13] and recently reviewed in [9]), which are the more commonly considered complete set of invariants of the string, commute with all the constraints and hence generate physical states when acting on the worldsheet vacuum, a consistently quantized version of the Pohlmeyer invariants should send boundary states of bare D-branes to those involving the excitation of a gauge field.

Indeed, up to a certain condition on the gauge field, this turns out to be true and works as follows:

If $X^\mu(\sigma)$ and $P_\mu(\sigma)$ are the canonical coordinates and momenta of the bosonic string, then $\mathcal{P}^\mu_\pm(\sigma) := \frac{1}{\sqrt{2T}}(P_\mu(\sigma) \pm T\eta_{\mu\nu}X'^\nu(\sigma))$, (where T is the string's tension and a prime denotes the derivative with respect to σ) are the left- and right-moving bosonic worldsheet fields for flat Minkowski background (in CFT context denoted by ∂X and $\bar\partial X$) and for any given constant gauge field A on target space the objects

$$W^{\mathcal{P}}_\pm[A] := \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\, A\cdot\mathcal{P}_\pm(\sigma)\right) \tag{1.1}$$

(where Tr is the trace in the given representation of the gauge group's Lie algebra and P denotes path-ordering along σ) Poisson-commute with all Virasoro constraints. In fact the coefficients of $\mathrm{Tr}(A^n)$ in these generalized Wilson lines do so seperately, and these are usually addressed as the *Pohlmeyer invariants*, even though we shall use this term for the full object (1.1).

Fundamentally, the reason for this invariance is just the reparameterization invariance of the Wilson line, which can be seen to imply that (1.1) remains unchanged under a substitution of $\mathcal{P}$ with a reparameterized version of this field. In [9] it was observed that an interesting example for such a substitution is obtained by taking the ordinary DDF oscillators

$$A^\mu_m \propto \int_0^{2\pi} d\sigma\; \mathcal{P}^\mu_-(\sigma)e^{im\frac{4\pi T}{k\cdot p}\,k\cdot X_-(\sigma)} \tag{1.2}$$

(where k is a lightlike vector on target space, X_- is the left-moving component of X, p is the center-of-mass momentum, and an analogous expression exists for $\mathcal{P}_+$) and forming "quasi-local" invariants

$$\mathcal{P}^{R\mu}_-(\sigma) := \frac{1}{\sqrt{2\pi}}\sum_{m=0}^{\infty} A^\mu_m e^{im\sigma} \tag{1.3}$$

from them.[1]

[1]We dare to use the same symbol A for the gauge field and for the DDF oscillators in order to comply with established conventions. The DDF oscillators will always carry a mode index m, however, and it should always be clear which object is meant.

One finds

$$W^{\mathcal{P}}[A] = W^{\mathcal{P}^R}[A] \tag{1.4}$$

and since the quantization of the $\mathcal{P}^R$ in terms of DDF oscillators is well known, this gives a consistent quantization of the Pohlmeyer invariants. This is the quantization that we shall use here to study boundary states.

The above construction has a straightforward generalization to the superstring and this is the context in which the relation between the Pohlmeyer invariants and boundary states turns out to have interesting aspects, (while the bosonic case follows as a simple restriction, when all fermions are set to 0).

So we consider the supersymmetric extension of (1.2), which, by convenient abuse of notation, we shall also denote by A^μ_m:

$$A^\mu_m \propto \int_0^{2\pi} d\sigma \left(\mathcal{P}_-(\sigma) + im\frac{\pi\sqrt{2T}}{k\cdot p} k\cdot\Gamma_-(\sigma)\,\Gamma^\mu_-(\sigma)\right) e^{im\frac{4\pi T}{k\cdot p}\,k\cdot X_-(\sigma)}\,, \tag{1.5}$$

where $\Gamma_\pm(\sigma)$ denote the fermionic superpartners of $\mathcal{P}_\pm$. From these we build again the objects (1.3) and finally $W^{\mathcal{P}^R}[A]$, which we address as the *super-Pohlmeyer* invariants.

Being constructed from the supersymmetric invariants $\mathcal{P}^R$, which again are built from (1.5), these manifestly commute with all of the super-Virasoro constraints. But in order to relate them to boundary states they need to be re-expressed in terms of the plain objects $\mathcal{P}$ and Γ. This turns out to be non-trivial and has some interesting aspects to it.

After these peliminaries we can state the first result to be reported here, which is

1. that on that subspace $\mathbf{P}_k$ of phase space where $k\cdot X_-$ is invertible as a function of σ (a condition that plays also a crucial role for the considerations of the bosonic DDF/Pohlmeyer relationship as discussed in [9]) the super-Pohlmeyer invariants built from (1.5) are equal to

$$W^{\mathcal{P}^R}[A]\Big|_{\mathbf{P}_k} = \mathrm{Tr\,P}\exp\left(\int_0^{2\pi} d\sigma\left(iA_\mu + [A_\mu, A_\nu]\,\frac{k\cdot\Gamma\,\Gamma^\nu}{2k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right), \tag{1.6}$$

2. that this expression extends to an invariant on all of phase space precisely if the transversal components of A mutually commute,

3. and that in this case the above is equal to

$$Y[A] := \mathrm{Tr\,P}\exp\left(\int_0^{2\pi} d\sigma\left(iA_\mu\mathcal{P}^\mu + \frac{1}{4}[A_\mu, A_\nu]\,\Gamma^\mu\Gamma^\nu\right)\right). \tag{1.7}$$

The second result concerns the application of the quantum version of these observables to the bare boundary state $|\mathrm{D9}\rangle$ of a space-filling D9-brane (see for instance appendix A of [3] for a brief review of boundary state formalism and further literature). Denoting by $\mathcal{E}^\dagger(\sigma) = \frac{1}{2}(\Gamma_+(\sigma) + \Gamma_-(\sigma))$ the differential forms on loop space (cf. section 2.3.1 of [3] and section 2.2 of [14] for the notation and nomenclature used here, and see [15] for a

JHEP10(2004)035

more general discussion of the loop space perspective) we find that for the above case of commuting transversal A the application of (1.7) to $|\mathrm{D9}\rangle$ yields

$$\begin{aligned}
&\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\,\left(iA_\mu \mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\nu}\Gamma^\mu\Gamma^\nu\right)\right)|\mathrm{D9}\rangle = \\
&\qquad = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\,\left(-i\sqrt{\frac{T}{2}}A_\mu X'^\mu + \frac{1}{4}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right)|\mathrm{D9}\rangle\,. \qquad (1.8)
\end{aligned}$$

which is, on the right hand side, precisely the boundary state describing a non-abelian gauge field on the D9 brane [1, 3] (for comparison one should rescale A as discussed in (2.42) below).

In summary this shows that and under which conditions the application of a quantized super-Pohlmeyer invariant to the boundary state of a bare D9 brane produces the boundary state describing a non-abelian gauge field excitation.

The structure of this paper closely follows the above outline:

First of all section 2.1 (p. 4) is concerned with the classical super-Pohlmeyer invariants and their expression in terms of local fields. Then section 2.1 (p. 4) discusses their cousins, the invariants of the general form (1.7). Both are related in section 2.3 (p. 8).

Then the quantization of the super-Pohlmeyer invariants is started in section 2.4 (p. 10). After an intermediate result concerning an operator ordering issue is treated in section 2.5 (p. 11) the quantum Pohlmeyer invariants are finally applied to the bare boundary state in section 2.6 (p. 13).

Section 3 (p. 15) gives some concluding remarks.

2. DDF operators, Pohlmeyer invariants and boundary states

2.1 Super-Pohlmeyer invariants

In [9] it was shown how from the classical DDF oscillators of the bosonic string one can construct *quasilocal* fields $\mathcal{P}^R$, which (Poisson-)commute with all the constraints and which, when used in place of X' in a Wilson line of a constant gauge field along the string, reproduce the Pohlmeyer invariants. It was mentioned that using the DDF oscillators of the superstring in this procedure leads to a generalization of the Pohlmeyer invariants to the superstring. Here we will work out the explicit form of the *super-Pohlmeyer invariants* obtained this way and point out that they are interesting in their own right.

Using the notation of [9] we denote by $\mathcal{P}^\mu(\sigma)$ the classical canonical left- or right-moving bosonic fields on the string, and by $\Gamma^\mu(\sigma)$ their fermionic partners, where the relation to the usual CFT notation is $\mathcal{P}^\mu \propto \partial X^\mu$ and $\Gamma^\mu \propto \psi^\mu$.

Our normalization is chosen such that the graded Poisson-brackets read

$$\begin{aligned}
[\Gamma^\mu(\sigma), \Gamma^\nu(\kappa)] &= -2\eta^{\mu\nu}\delta(\sigma-\kappa) \\
[\mathcal{P}^\mu(\sigma), \mathcal{P}^\nu(\kappa)] &= -\eta^{\mu\nu}\delta'(\sigma-\kappa)\,. \qquad (2.1)
\end{aligned}$$

The classical bosonic DDF oscillators A^μ_m of the superstring are obtained by acting with the supercharge

$$G_0 = \frac{i}{\sqrt{2}} \int d\sigma\, \Gamma^\mu \mathcal{P}_\mu \tag{2.2}$$

(we concentrate on the Ramond sector for notational simplicity) on qintegrals over weight 1/2 fields:

$$\begin{aligned} A^\mu_m &:= \left[G_0, \frac{i}{\sqrt{4\pi}} \oint d\sigma\; \Gamma^\mu e^{-imR} \right] \\ &= \frac{1}{\sqrt{2\pi}} \oint d\sigma \left(\mathcal{P}^\mu + im \frac{\pi\sqrt{2T}}{k\cdot p} k\cdot \Gamma\Gamma^\mu \right) e^{-imR}\,, \end{aligned} \tag{2.3}$$

where

$$R(\sigma) := -\frac{4\pi T}{k\cdot p} k\cdot X_\pm(\sigma) \tag{2.4}$$

and $p^\mu = \int_0^{2\pi} P^\mu(\sigma)$.

By construction, the A^μ_m super-Poisson-commute with all the constraints. From the A^μ_m quasi-local objects $\mathcal{P}^R$ are reobtained by Fourier transforming from the integral mode index m to the parameter σ:

$$\begin{aligned} \mathcal{P}^R(\sigma) &:= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} A_n\, e^{in\sigma} \\ &= \int_0^{2\pi} d\tilde\sigma \left(\mathcal{P}(\tilde\sigma)\, \delta\left(R(\tilde\sigma) - \sigma\right) + \frac{i\pi\sqrt{2T}}{k\cdot p} k\cdot\Gamma(\tilde\sigma)\,\Gamma(\tilde\sigma)\, \frac{\partial}{\partial\sigma}\delta\left(R(\tilde\sigma)-\sigma\right)\right). \end{aligned} \tag{2.5}$$

The role of the DDF oscillators played here is the derivation of this expression. Their invariance was rather easy to enforce and check, but by taking combinations of them as in (2.5) and further constructions below, we can now build objects which are necessarily still invariants, but whose invariance is much less obvious.

Since the DDF oscillators A^μ_m won't be explicitly needed anymore in the following, we take the liberty to reserve the letter A from now on to describe a gauge connection on target space. We shall be interested in the Wilson line

$$W^{\mathcal{P}^R}[A] := \mathrm{Tr}\,\mathrm{P}\exp\left(i \int_0^{2\pi} d\sigma\; A\cdot \mathcal{P}^R(\sigma) \right) \tag{2.6}$$

with respect to this gauge connection A, constructed using the "generalized tangent vector" $\mathcal{P}^R$ which plays the role of the true tangent vector X' found in ordinary Wilson lines. Because this object follows in spirit closely the construction principle of the bosonic Pohlmeyer invariants, and because its bosonic component coincides with the purely bosonic Pohlmeyer invariant, we shall here address it as the *super-Pohlmeyer invariant.* In the following a form of this object in terms of the original local fields $\mathcal{P}$ and Γ is derived, which will illuminate its relation to supersymmetric boundary states.

JHEP10(2004)035

The integrand of (2.6) can be put in a more insightful form by means of a couple of manipulations:

Following the development in [9] (cf. eq. (2.43)) we now temporarily restrict attention to the subspace $\mathbf{P}_k$ of phase space on which the function R is invertible, in which case it is, by construction, 2π-periodic. On that part of phase space (and only there) the integral in (2.5) can be evaluated to yield

$$\mathcal{P}^R(\sigma)\big|_{\mathbf{P}_k} = \left(R^{-1}\right)'(\sigma)\mathcal{P}\left(R^{-1}(\sigma)\right) + \frac{\pi i\sqrt{2T}}{k\cdot p}\frac{\partial}{\partial\sigma}\left(\left(R^{-1}\right)'(\sigma)k\cdot\Gamma\left(R^{-1}(\sigma)\right)\Gamma\left(R^{-1}(\sigma)\right)\right). \tag{2.7}$$

The first term is known from the bosonic theory (eq. (2.51) in [9]). The second term involves the fermionic correction due to supersymmetry, and its remarkable property is that it is a total σ-derivative. This means that when $\mathcal{P}^R$ is inserted in a multi-integral as they appear in (2.6), the fermionic term will produce boundary terms and hence coalesce with neighbouring integrands.

Before writing this down in more detail first note that due to k being a null vector the fermionic terms can never coalesce with themselves, because of

$$\begin{aligned}
&\left(R^{-1}\right)'k\cdot\Gamma\left(R^{-1}\right)A\cdot\Gamma\left(R^{-1}\right)(\sigma)\frac{\partial}{\partial\sigma}\left(\left(R^{-1}\right)'k\cdot\Gamma\left(R^{-1}\right)A\cdot\Gamma\left(R^{-1}\right)\right)(\sigma) = \\
&= \frac{1}{2}\frac{\partial}{\partial\sigma}\underbrace{\left(\left(R^{-1}\right)'k\cdot\Gamma\left(R^{-1}\right)A_\mu\Gamma^\mu_-\left(R^{-1}\right)\right)^2(\sigma)}_{=0} \\
&= 0\,.
\end{aligned} \tag{2.8}$$

This vanishing result depends on the Grassmann properties of the classical fermions Γ, which we are dealing with here. The generalization of the present development to the quantum theory requires more care and is dealt with below.

Using (2.8) a little reflection shows that, when the total derivative terms in (2.6) are all integrated over and coalesced at the integration bounds with the neighbouring terms $iA\cdot\mathcal{P}$, this yields

$$\begin{aligned}
&\mathrm{Tr}\,\mathrm{P}\exp\left(i\int_0^{2\pi}d\sigma\,A\cdot\mathcal{P}^R(\sigma)\right)\bigg|_{\mathbf{P}_k} = \\
&= \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi}d\sigma\left(iA_\mu + [A_\mu, A_\nu]\,\frac{\pi\sqrt{2T}}{k\cdot p}(R^{-1})'(\sigma)k\cdot\Gamma(R^{-1}(\sigma))\Gamma^\nu(R^{-1}(\sigma))\right)\times\right. \\
&\qquad\left.\times\,(R^{-1})'(\sigma)\mathcal{P}^\mu\left(R^{-1}(\sigma)\right)\right).
\end{aligned} \tag{2.9}$$

This expression simplifies drastically when a change of variable $\tilde\sigma := R^{-1}(\sigma)$ is performed in the integral, as in (2.23) of [9]:

$$\cdots = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi}d\tilde\sigma\,\left(iA_\mu + [A_\mu, A_\nu]\,\frac{\pi\sqrt{2T}}{k\cdot p}(R^{-1})'\left(R\left(\tilde\sigma\right)\right)k\cdot\Gamma\left(\tilde\sigma\right)\,\Gamma^\nu\left(\tilde\sigma\right)\right)\mathcal{P}^\mu\left(\tilde\sigma\right)\right). \tag{2.10}$$

The fermionic term further simplifies by using $(R^{-1})'(R(\tilde{\sigma})) = 1/R'(\tilde{\sigma})$ and then equation (2.42) of [9], which gives $(R^{-1})'(R(\tilde{\sigma})) = \frac{k\cdot p}{2\pi\sqrt{2T}}\frac{1}{k\cdot\mathcal{P}}$. This way the above is finally rewritten as

$$W^{\mathcal{P}^R}[A]\Big|_{\mathbf{P}_k} = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu + [A_\mu, A_\nu]\,\frac{k\cdot\Gamma\,\Gamma^\nu}{2k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right). \tag{2.11}$$

This is the advertized explicit form of the super-Pohlmeyer invariant in terms of local fields, when restricted to $\mathbf{P}_k$.

The right hand side extends to an observable on all of phase space in the obvious way and it is of interest to study if this extension is still an invariant. This is the content of the following subsections.

2.2 Another supersymmetric extension of the bosonic Pohlmeyer invariants

We address the objects (2.6) as super-Pohlmeyer invariants, because they are obtained from the bosonic Pohlmeyer invariants written in the form $\mathrm{Tr}\,\mathrm{P}\exp(\int_0^{2\pi} d\sigma\, A\cdot\mathcal{P}^R(\sigma))$ of equation (2.52) of [9] by replacing the bosonic *quasi-local* invariants $\mathcal{P}^R$ by their supersymmetric version (2.5). In this sense this supersymmetric extension is *local*, or rather "quasi-local", since the $\mathcal{P}^R$ are. But it turns out that there is another fermionic extension of the bosonic Pohlmeyer invariant $\mathrm{Tr}\,\mathrm{P}\exp(\int_0^{2\pi} d\sigma\ A\cdot\mathcal{P}(\sigma))$ which Poisson-commutes with all the super-Virasoro generators, and which is not local in this sense, namely

$$Y[A] := \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu\mathcal{P}^\mu(\sigma) + \frac{1}{4}[A_\mu, A_\nu]\,\Gamma^\mu(\sigma)\Gamma^\nu(\sigma)\right)\right). \tag{2.12}$$

Here the integrand itself does not Poisson-commute with the supercharge G_0, but $Y[A]$ as a whole does. (This can easily be generalized even to non-constant A, but we will here be content with writing down all expression for the case of constant A. Non-constant A will be discussed in the context of the quantum theory further below.)

Invariance under the bosonic Virasoro generators is immediate, because the integrand has unit weight. All that remains to be checked is hence

$$[G_0, Y[A]] = 0\,. \tag{2.13}$$

Proof. This is best seen by following the logic involved in the derivation of equation (3.8) in [3]: There are terms coming from $[G_0, iA\cdot\mathcal{P}(\sigma)] \propto iA\cdot\Gamma'(\sigma)$ which coalesce at the integration boundary with $iA\cdot\mathcal{P}$ to give $-[A_\mu, A_\nu]\,\Gamma^\mu\mathcal{P}^\nu$. This cancels with the contribution from $\left[G_0, \frac{1}{4}[A_\mu, A_\nu]\,\Gamma^\mu\Gamma^\nu\right] \propto [A_\mu, A_\nu]\,\Gamma^\mu\mathcal{P}^\nu$. (Here we write $\propto$ only as a means to ignore the irrelevant global prefactor $i/\sqrt{2}$ in (2.2).) Moreover, there is coalescence of $A\cdot\Gamma'$ with $[A_\mu, A_\nu]\,\Gamma^\mu\Gamma^\nu$ which yields $[A_\kappa, [A_\mu, A_\nu]]\,\Gamma^\kappa\Gamma^\mu\Gamma^\nu = 0$, so that everything vanishes. This establishes the full invariance of $Y[A]$ under the super-Virasoro algebra. $\square$

With this insight in hand, one can make a curious observation. Write $A_+ := k\cdot A$ and consider the special case where all transversal components of A together with the additional lightlike component A_- mutually commute

$$[A_i, A_j] = 0\,, \qquad \forall\, i,j \neq +\,. \tag{2.14}$$

Then

$$[A_\mu, A_\nu] \frac{k \cdot \Gamma\Gamma^\nu}{2k \cdot \mathcal{P}} \mathcal{P}^\mu = \frac{1}{2} [A_+, A_i] \Gamma^+ \Gamma^i$$
$$= \frac{1}{4} [A_\mu, A_\nu] \Gamma^\mu \Gamma^\nu \,. \tag{2.15}$$

Comparison of (2.11) with (2.12) hence shows that in this case the super-Pohlmeyer invariant (2.11) and the invariant (2.12) coincide:

$$[A_i, A_j] = 0 \,, \qquad \forall\, i, j \neq + \Rightarrow \; W^{\mathcal{P}^R}[A]\Big|_{\mathbf{P}_k} = Y[A] \,. \tag{2.16}$$

So in particular in the case (2.14) the extension of the right hand side of (2.11) to all of phase space is still an invariant.

Comparing (2.11) with equation (3.14) of [3] it is obvious, and will be discussed in more detail below, that $Y[A]$ must somehow be closely related to the boundary deformation operator describing non-abelian A-field excitations. Together with (2.16) this gives a first indication of how super-Pohlmeyer invariants give insight into boundary states of the superstring.

Before discussing this in more detail the next section investigates the most general condition under which the extension of the right hand side of (2.11) to all of phase space is still an invariant. It turns out that there are other cases besides (2.14).

2.3 Invariance of the extension of the restricted super-Pohlmeyer invariants

For the bosonic string the constraint $\mathcal{P} \cdot \mathcal{P} = 0$, which says that $\mathcal{P}$ is a null vector in target space, ensured that the invertibility of R was preserved by the evolution generated by the constraints (cf. the discussion on p. 12 of [9]).

The same is no longer true for the superstring, where we schematically have $\mathcal{P} \cdot \mathcal{P} + \Gamma \cdot \Gamma' = 0$, instead. It follows that we cannot expect the extension of the right hand side of (2.11) to all of (super-)phase space to super-Poisson commute with all the constraints, since the flow induced by the constraints will in general leave the subspace $\mathbf{P}_k$. Only for the bosonic string does the flow induced by the constraints respect $\mathbf{P}_k$.

Notice that this is not in contradiction to the above result that on $\mathbf{P}_k$ the super-Pohlmeyer invariant (2.6) (which by construction super-Poisson commutes with all the constraints) coincides with (2.11). Two functions which conincide on a subset of their mutual domains need not have coinciding derivatives at these points.

First of all one notes that the invariance under the action of the bosonic constraints is still manifest in (2.11). Because the integrand still has unit weight one checks this simply by using the same reasoning as in equation (2.19) of [9].

But the result of super-Poisson commuting with the supercharge G_0 is rather non-obvious. A careful calculation shows that the result vanishes if and only if

$$[A_i, A_j] = 0 \,, \qquad \forall\, i, j \neq + \tag{2.17}$$

or

$$k \cdot \Gamma' = 0 = k \cdot \mathcal{P}' \,. \tag{2.18}$$

The first condition is that already discussed in section 2.2. The second condition is nothing but the defining condition of *lightcone gauge* on the worldsheet.

Notice that these two conditions are very different in character. When the first (2.17) is satisfied it means that the extension of the right hand side of (2.11) to all of phase space is indeed an honest invariant. When the first condition is not satisfied then the extenstion of the right hand side of (2.17) to all of phase space is simply not an invariant. Still, it is an object whose Poisson-commutator with the super-Virasoro constraints vanishes on that part of phase space where (2.18) holds.

We now conclude this subsection by giving the detailed *proof* for the above two conditions.

Proof. First consider the terms of fermionic grade 1. These are contributed by

$$[G_0, A \cdot \mathcal{P}] \propto A \cdot \Gamma' \tag{2.19}$$

as well as

$$[A_\mu, A_\nu] \, \frac{[G_0, k \cdot \Gamma] \, \Gamma^\nu \mathcal{P}^\mu}{2k \cdot \mathcal{P}} \propto -[A_\mu, A_\nu] \, \Gamma^\nu \mathcal{P}^\mu \,. \tag{2.20}$$

The other remaining fermionic contraction does not contribute, due to

$$[A_\mu, A_\nu] \, [G_0, \Gamma^\nu] \, \mathcal{P}^\mu \propto -2 \, [A_\mu, A_\nu] \, \mathcal{P}^\nu \mathcal{P}^\mu = 0 \,. \tag{2.21}$$

In the path ordered integral the terms (2.19) appear as

$$\begin{aligned} \cdots iA \cdot \mathcal{P} \, (\sigma_{i-1}) \int_{\sigma_{i-1}}^{\sigma_{i+1}} iA \cdot \Gamma' \, (\sigma_i) \; d\sigma_i \; iA \cdot \mathcal{P} \, (\sigma_{i+1}) \cdots = \\ = \cdots \Big(\, (A \cdot \mathcal{P} A \cdot \Gamma) \, (\sigma_{i-1}) \, iA \cdot \mathcal{P} \, (\sigma_{i+1}) - \\ - \, iA \cdot \mathcal{P} \, (\sigma_{i-1}) \, (A \cdot \Gamma A \cdot \mathcal{P}) \, (\sigma_{i+1}) \Big) \cdots \,. \end{aligned} \tag{2.22}$$

(This is really a special case of the general formula (3.8) in [3].) This way the term

$$iA_\mu \Gamma^\mu \, iA_\nu \mathcal{P}^\nu - iA_\mu \mathcal{P}^\nu \, iA_\mu \Gamma^\mu \, = [A_\mu, A_\nu] \, \Gamma^\nu \mathcal{P}^\mu \tag{2.23}$$

is produced, and it cancels precisely with (2.20).

This verifies that there are no terms of grade 1.

Now consider the remaining terms of grade 3. It is helpful to write

$$[A_\mu, A_\nu] \, \frac{k \cdot \Gamma \Gamma^\nu}{2k \cdot \Gamma} \mathcal{P}^\mu = = \frac{1}{2} \, [A_+, A_i] \, \Gamma^+ \Gamma^i + [A_i, A_j] \, \frac{k \cdot \Gamma \Gamma^j}{2k \cdot \mathcal{P}} \mathcal{P}^i \,. \tag{2.24}$$

The first term on the right hand side gives nothing of grade 3 when Poisson-commuted with G_0. The second term however gives rise to

$$\begin{aligned} &\left[G_0, [A_i, A_j] \, \frac{k \cdot \Gamma \, \Gamma^j}{2k \cdot \mathcal{P}} \mathcal{P}^i \right] = \\ &= [A_i, A_j] \left(\frac{k \cdot \Gamma \, \Gamma^j}{2k \cdot \mathcal{P}} \Gamma'^i - \frac{k \cdot \Gamma \, \Gamma^j}{2(k \cdot \mathcal{P})^2} k \cdot \Gamma' \right) + \text{terms already considered} \\ &= [A_i, A_j] \left(\frac{k \cdot \Gamma}{4k \cdot \mathcal{P}} (\Gamma^j \Gamma^i)' - \frac{k \cdot \Gamma \, \Gamma^j}{2(k \cdot \mathcal{P})^2} k \cdot \Gamma' \right) + \text{terms already considered} \\ &= [A_i, A_j] \left(\frac{k \cdot \Gamma}{4k \cdot \mathcal{P}} \Gamma^j \Gamma^i \right)' + \alpha + \text{terms already considered} \,, \end{aligned} \tag{2.25}$$

where we have abbreviated with

$$\alpha := -[A_i, A_j]\left(\left(\frac{k\cdot\Gamma}{2k\cdot\mathcal{P}}\right)'\Gamma^j\Gamma^i + \frac{k\cdot\Gamma\,\Gamma^j}{2(k\cdot\mathcal{P})^2}k\cdot\Gamma'\right) \tag{2.26}$$

two terms which will *not* cancel with anything in the following. (Notice that they are proportional to σ-derivatives of longitudinal objects (along k).)

The remaining first term on the right hand side of (2.25) coalesces with $iA_+\mathcal{P}^+$ to yield $\frac{i}{4}[A_+, [A_i, A_j]]\,\Gamma^+\Gamma^i\Gamma^j$. This cancels against the coalescence of (2.19) with the first term on the right hand side of (2.24) which gives the term $\frac{i}{2}[A_j, [A_+, A_i]]\,\Gamma^j\Gamma^+\Gamma^i$, because together they become the longitudinal component of the exterior covariant derivative of the field strength of A, which vanishes. The transversal component of this exterior derivative of the field strength appears in the remaining terms:

First there is the remaining coalescence of (2.19) with the second term on the right hand side of (2.24), which yields $i\,[A_k, [A_i, A_j]]\,\frac{k\cdot\Gamma}{2k\cdot\mathcal{P}}\Gamma^k\Gamma^j\mathcal{P}^i$. Together with the remaining coalescence of the first term on the right of (2.25) with the transversal $iA_j\mathcal{P}^j$ which gives rise to $i\,[A_k, [A_i, A_j]]\,\frac{k\cdot\Gamma}{4k\cdot\mathcal{P}}\Gamma^i\Gamma^j\mathcal{P}^k$ one gets something proportional to

$$\left[G_0, \underbrace{[A_k, [A_i, A_j]]\,\Gamma^k\Gamma^i\Gamma^j}_{=0}\right] = 0\,,$$

which vanishes because it involves the transversal part of the gauge covariant exterior derivative of the field strength of A.

In summary, the only terms that remain are those of (2.26). When the σ-derivative is written out this are three terms which have to vanish seperately, because they contain different combinations of fermions. Clearly they vanish precisely if (2.17) or (2.18) are satisfied. This completes the proof. $\square$.

2.4 Quantum super-Pohlmeyer invariants

The DDF-invariants (2.3) are, as discussed in equation (2.12) of [9], still invariants after quantization in terms of DDF oscillators. If we take the liberty to denote the quantized objects $\mathcal{P}$ and Γ by the same symbols as their classical counterparts, then the only thing that changes in the notation of the above sections is that the canonical super-commutation relations (2.1) pick up an imaginary factor

$$[\mathcal{P}^\mu(\sigma), \mathcal{P}(\kappa)] = -i\eta^{\mu\nu}\delta'(\sigma-\kappa)\ . \tag{2.27}$$

This again introduces that same factor in the second term of (2.3) and similarly in the following expressions.

The quantization of the super-Pohlmeyer invariant (2.6) is a trivial consequence of the quantization of the DDF invariants that it is built from, and, with that imaginary unit taken care of, its restriction (2.11) to the case where R is invertible reads

$$\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\left(iA_\mu + \frac{i}{2}[A_\mu, A_\nu]\frac{k\cdot\Gamma\Gamma^\nu}{k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right). \tag{2.28}$$

Noting that our A is taken to be hermitian and that hence the gauge field strength is

$$\begin{aligned} F_A &= -i(d+iA)^2 \\ &= dA + iA \wedge A \\ &= \left(\partial_{[\mu} A_{\nu]} + \frac{i}{2}\left[A_\mu, A_\nu\right]\right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}(F_A)_{\mu\nu} dx^\mu \wedge dx^\nu \end{aligned} \tag{2.29}$$

the second term in the integrand is related to the field strength as in

$$\cdots = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu + \frac{1}{2}(F_A)_{\mu\nu}\frac{k\cdot\Gamma\Gamma^\nu}{k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right). \tag{2.30}$$

In the case $[A_i, A_j] = 0$ (2.14) we hence obtain the quantized version of (2.12) in the form

$$Y[A] = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu\mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\mu}\Gamma^\mu\Gamma^\nu\right)\right). \tag{2.31}$$

While the quantized super-Pohlmeyer invariant, being constructed from invariant DDF operators, is itself a quantum invariant in that it commutes with all the super-Virasoro constraints, the proof in section 2.3 of the invariance condition of the restricted and then extended form (2.11) receives quantum corrections. In its classical version the proof makes use of the Grassmann property of the fermions Γ. Quantumly there will be diverging contractions in products of Γs which not only prevent the application of the proof to the quantum theory but also make the expression (2.28) ill defined without some regularization prescription. As we discuss in the conclusion section 3 these regularizations can be done, but instead of applying them here we will make contact to the approach which was pioneered in [11, 12] and generalized to the nonabelian case in [3], where generalized Wilson lines as above are applied without regularization to boundary states and the vanishing of divergences in the result is then shown to be equivalent to the equations of motion of the background fields.

This application of (2.31) to a bare boundary states is the content of section 2.6. But before coming to that a technicality needs to be discussed, which is done in the next section.

2.5 On an operator ordering issue in Wilson lines along the closed string

For applying a generalized Wilson line of the kind discussed above to any string state, it is helpful to understand how the operators in the Wilson line can be commuted past each other to act on the state on the right. It turns out that under a certain condition, which is fulfilled in the cases we are interested in, the operators can be freely commuted. This works as follows:

A generalized Wilson line of the form

$$W^{\mathcal{P}}[A] = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} A\cdot\mathcal{P}(\sigma)\,d\sigma\right) \tag{2.32}$$

with even graded $\mathcal{P}$ (which could be the $\mathcal{P}$ or $\mathcal{P}^R$ of the previous sections but also more general objects) breaks up like

$$W^{\mathcal{P}}[A] = \sum_{n=0}^{\infty} Z^{\mu_1\cdots\mu_n}\,\mathrm{Tr}\,(A_{\mu_1}\cdots A_{\mu_2}) \tag{2.33}$$

into iterated integrals

$$\begin{aligned} Z^{\mu_1\cdots\mu_N} &= \frac{1}{N}\Bigg[\int_{0<\sigma^1<\sigma^2<\cdots<\sigma^N<2\pi} d^N\sigma + \int_{0<\sigma^N<\sigma^1<\cdots<\sigma^{N-1}<2\pi} d^N\sigma + \\ &\quad + \int_{0<\sigma^{N-1}<\sigma^N<\cdots<\sigma^{N-2}<2\pi} d^N\sigma + \cdots\Bigg]\mathcal{P}^{\mu_1}(\sigma^1)\cdots\mathcal{P}^{\mu_N}(\sigma^N)\,. \end{aligned} \tag{2.34}$$

In equation (2.17) of [9] if was noted that the integration domain can equivalently be written as

$$Z^{\mu_1\cdots\mu_N} = \frac{1}{N}\int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \cdots \int_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N \mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\cdots\mathcal{P}^{\mu_N}(\sigma^N)\,. \tag{2.35}$$

This is seen by simply replacing all $\sigma^i < \sigma^1$ for $i > 1$ by $\sigma^i + 2\pi$. Due to the periodicity of $\mathcal{P}$ this does not change the value of the integral but yields the integration bounds used in (2.35).

The reason why this is recalled here is that a slight generalization of this fact will be needed in the following. Namely for any integer M with $1 < M < N$ one can obviously more generally write

$$\begin{aligned} Z^{\mu_1\cdots\mu_N} &= \int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \int_{\sigma^1}^{\sigma^M} d\sigma^2 \times \\ &\quad \times\cdots \int_{\sigma^{M-2}}^{\sigma^M} d\sigma^{M-1} \int_{\sigma^M}^{\sigma^1+2\pi} d\sigma^{M+1} \times \\ &\quad \times\cdots \int_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N \mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\cdots\mathcal{P}^{\mu_N}(\sigma^N)\,. \end{aligned} \tag{2.36}$$

Equation (2.35) follows as the special case with $M = 2$.

The motivation for these considerations is the following:

Classically, the $\mathcal{P}$ commute among each other. Therefore the ordering of the $\mathcal{P}$ in the integrand makes no difference, only the combination of spacetime index μ_i with integration variable σ^i does.

Here we want to note that this remains true at the quantum level *if*

$$[\mathcal{P}(\sigma), \mathcal{P}(\kappa)] \propto \delta'(\sigma - \kappa)\,. \tag{2.37}$$

This is readily seen by commuting $\mathcal{P}(\sigma^1)$ with $\mathcal{P}(\sigma^M)$ in (2.36). The result has the form

$$\int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \delta'(\sigma^1 - \sigma^M) F(\sigma^1, \cdots, \sigma^N) = \int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \delta\left(\sigma^1 - \sigma^\kappa\right) \times$$

$$
\begin{aligned}
&\times \frac{\partial}{\partial \sigma^M} F\left(\sigma^1, \cdots, \sigma^N\right) \\
&= \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \delta\left(\sigma^1 - \sigma^\kappa\right) \times \\
&\times \frac{\partial}{\partial \sigma^M} F\left(\sigma^1, \cdots, \sigma^N\right) \\
&= 0\,,
\end{aligned}
\tag{2.38}
$$

so that all resulting commutator terms vanish. Every other commutator can be obtained by using the cyclic invariance in the integration variables.

More generally, any two (even graded, periodic) objects $A(\sigma)$, $B(\kappa)$ in the integrand of an iterated integral of the form (2.34) whose commutator is proportional to $[A(\sigma), B(\kappa)] \propto \delta'(\sigma - \kappa)$ can be commuted past each other in the Wilson line without affecting the value of the integral.

This simple but crucial observation will be needed below for the demonstration that Pohlmeyer-invariants map the boundary state of a bare D-brane to that describing a brane with a nonabelian gauge field turned on.

2.6 Super-Pohlmeyer and boundary states

We now have all ingredients in place to apply the super-Pohlmeyer invariant to the boundary state of a bare D9 brane. A brief review of the idea of boundary states adapted to the present context is given in [3], but in fact only two simple relations are needed for the following:

If $|\mathrm{D9}\rangle$ is the boundary state of the space-filling BPS D9 brane, then (due to equation (2.26) in [9] and section 2.3.1 in [3]) we have

$$
\mathcal{P}^\mu(\sigma)\,|\mathrm{D9}\rangle = \sqrt{\frac{T}{2}} X'^\mu(\sigma)\,|\mathrm{D9}\rangle
\tag{2.39}
$$

and

$$
\Gamma^\mu(\sigma)\,|\mathrm{D9}\rangle = \mathcal{E}^{\dagger\mu}(\sigma)\,|\mathrm{D9}\rangle\,.
\tag{2.40}
$$

Using the results of section 2.5 such a replacement extends to the full Wilson line made up from these objects:

Consider the extension (2.31) of the restricted super-Pohlmeyer invariant with $A_+ \neq 0$ and furthermore only mutually commuting spatial components of A nonvanishing. In this case the fermionic terms in the integrand have trivial commutators so that the integrand as a whole satisfies condition (2.37). Therefore, according to the result of section 2.5, we can move all appearances of $\mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\nu}\Gamma^\mu\Gamma^\nu$ to the boundary state $|\mathrm{D9}\rangle$ on the right, change it there to $\sqrt{\frac{T}{2}}X'^\mu + \frac{1}{4}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}$ and then move this back to the original position (noting that still $[X'(\sigma), \mathcal{P}(\kappa)] \propto \delta'(\sigma - \kappa)$). This way we have

$$
\begin{aligned}
&\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\, \left(iA_\mu \mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\nu}\Gamma^\mu\Gamma^\nu\right)\right)|\mathrm{D9}\rangle = \\
&\quad = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\, \left(-i\sqrt{\frac{T}{2}}A_\mu X'^\mu + \frac{1}{4}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right)|\mathrm{D9}\rangle\,.
\end{aligned}
\tag{2.41}
$$

If we allowed ourself to regulate all the generalized Wilson lines considered here by a point-splitting method as in [1], i.e. by taking care that no local fields in the Wilson line ever come closer than some samll distance σ, then the above step becomes a triviality. Indeed, the result of [1] together with those of [11, 12, 3] shows that this is a viable approach, because the condition for the ϵ-regularized Wilson line to be still an invariant is the same as that of the non-regularized Wilson line to be free of divergences and hence well defined. This is discussed further in section 3.

It will be convenient for our purposes to rescale A as

$$A \mapsto -\sqrt{\frac{2}{T}}A\,, \tag{2.42}$$

so that this becomes

$$\cdots = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu X'^\mu + \frac{1}{2T}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right)|\mathrm{D9}\rangle\,. \tag{2.43}$$

This is finally our main result, because this is precisely the boundary state of a nonabelian gauge field as considered in equation (3.14) of [3], which is a generalization of the abelian case studied in [11, 12]. The same form of the boundary state is obtained from equations (3.7), (3.8) in [1] when in the expression given there the integral over the Grassmann variables is performed (following the computation described on pp. 236-237 of [16]).

The boundary state (2.43) has two important properties:

1. **Super-Ishibashi property of the boundary state.** The defining property of boundary states is that they are annihilated by the generators $\mathcal{L}_K$ of σ-reparameterization as well as, in the superstring case, by their square root $\mathbf{d}_K$, which is a deformed exterior derivative on loop space. $\mathcal{L}_K$ is a linear combination of left- and right-moving bosonic super-Virasoro generators, while $\mathbf{d}_K$ is a combination of fermionic super-Virasoro generators, as discussed in [3].

   It is noteworthy that the state (2.43) indeed satisfies the Ishibashi conditions. Naively this must be the case, because this state is obtained from the bare $|\mathrm{D9}\rangle$, which does satisfy it by definition, by acting on it with a super-Pohlmeyer operator, that commutes with all constraints and hence leaves the Ishibashi property of $|\mathrm{D9}\rangle$ intact. But above we mentioned that the restricted form (2.28) of the quantized Pohlmeyer invariants that this state comes from has potential quantum anomalies which would spoil this invariance. These are due to the non-Grassmann property of the quantized fermions Γ. However, after application to the bare $|\mathrm{D9}\rangle$ which gives (2.43), the left- and right-moving fermions are replaced by their polar combination $\mathcal{E}^\dagger$, and these again enjoy the Grassmann property (they are nothing but differential forms on loop space). For this reason the final result can again enjoy the Ishibashi property, which means nothing but super-reparameterization invariance with respect to σ.

   Proof. The invariance under reparameterizations induced by $\mathcal{L}_K$ is manifest, analogous in all the cases considered here before, since (2.43) is the generalized Wilson line over an object of unit reparameterization weight.

JHEP10(2004)035

The only nontrivial part that hence needs to be checked is the commutation with $\mathbf{d}_K$ and here we only need to know that $[\mathbf{d}_K, X^\mu(\sigma)] = \mathcal{E}^{\dagger\mu}(\sigma)$.

Applying this to (2.43) we get, in the same manner as in the similar computations before, from $[\mathbf{d}_K, iA_\mu X'^\mu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathcal{E}^{\dagger\mu}X'^\nu + \left(iA_\mu \mathcal{E}^{\dagger\mu}\right)'$ coalesced terms $-\left[A_\mu, A_\nu\right]\mathcal{E}^{\dagger\mu}X'^\nu$ and $\frac{i}{2T}\left[A_\kappa, (F_A)_{\mu\nu}\right]\mathcal{E}^{\dagger\kappa}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}$ at the integration boundaries.

These combine with the terms $\left[\mathbf{d}_K, \frac{1}{2T}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right] = \frac{1}{2T}(\partial_{[\kappa}(F_A)_{\mu\nu]}\mathcal{E}^{\dagger\kappa}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}) + -i(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}X'^\nu$ to $\left(i(\partial_\mu A_\nu - \partial_\nu A_\mu) - [A_\mu, A_\nu] - i(F_A)_{\mu\nu}\right)\mathcal{E}^{\dagger\mu}X'^\nu = 0$ and $\frac{1}{2T}\left(\partial_{[\kappa}(F_A)_{\mu\nu} + i\left[A_{[\kappa}, (F_A)_{\mu\nu]}\right]\right)\mathcal{E}^{\dagger\kappa}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu} = 0$. Hence all terms vanish and $\mathbf{d}_K$ commutes with (2.43). $\square$

2. **Nonlinear gauge invariance of the boundary state.** A generic state constructed from gluon vertices for nonabelian A will generically not be invariant under a target space gauge transformation $A \to UAU^\dagger + U(dU^\dagger)$. The generalized Wilson line in (2.43) however does have this invariance - at least at the classical level. This follows from the general invariance properties of Wilson lines (for details see appendix B of [3]) and depends crucially on the appearance of the gauge covariant field strength F_A in (2.43).

Further comments on the nature of the boundary states considered here, in particular on the question of divergences and their regularization, are left to the concluding section.

3. Conclusion

The super-Pohlmeyer invariants, whose construction principle was only briefly mentioned at the end of [9], have been studied here in more detail. In particular their expression in terms of local fields has been worked out on that part of phase space where it exists. The extension of this expression to the full phase space has been found to be invariant on all of phase space if the transversal components of the gauge field mutually commute.

Quantizing the result and applying it to the boundary state of a bare D9 brane was shown to yield boundary states of the form considered before in [1, 3], which are straightforward non-abelian generalizations of those studied in [11].

This result shows that the Pohlmeyer invariants, which before have only appeared in the connection with vague hopes to circumvent well-known quantum effects like the critical dimension, do have a role to play in (standard) string theory.

While this might not appear as much of a surprise in light of the result [9] which showed that the Pohlmeyer invariants are a subset of all DDF invariants that have the crucial property of mapping physical states to physical states by commuting with all the (super-)Virasoro constraints, it seems noteworthy that it is a priori not obvious that this ability to generate the physical spectrum translates also to a generation of boundary states that satisfy the Ishibashi condition.

And indeed, it was shown above that a precise match between super-Pohlmeyer invariants and boundary states is manifest only in the special case where the transversal components of the gauge field mutually commute. On the other hand, we did not show that in other cases the result of acting with the super-Pohlmeyer invariant on the bare D9-brane boundary state does not produce a new some boundary, but it is at least not obvious.

Of course, naively it is obvious that the result of acting with a super-Pohlmeyer invariant on a state which satisfies the Ishibashi conditions still satisfies these conditions, simply due to the very invariance property of the Pohlmeyer invariants. But the result will in general need regularization and/or a condition on the gauge field that ensures vanishing of contact term divergences, and for the case where the transversal components of A do not mutually commute we did not show that the result has this property.

This is in contrast to the case where the transversal components do mutually commute, and where the lightlike component of A which is non-parallel to k vanishes. In this case we did show that the result of acting with the super-Pohlmeyer invariant associated with that A-field on $|D9\rangle$ does produce the result (2.43). And this result is known to be well defined if (and only if) A satisfies its background equations of motion.

This can either be seen by introducing a regularization and checking that this regularization preserves the invariance properties iff the background equations of motion hold, as was done in [1]. Or, alternatively, no regularization is used and the resulting divergences are shown to vanish when A satisfies its equations of motion. This was done for the abelian case to second order in [11, 12] and for the non-abelian case to first order in [3].

While this relation between quantum divergences and background equations of motion is perfectly natural in the context of boundary states, it is a new aspect in the study of Pohlmeyer invariants. It shows that even though these have a consistent quantization in terms of DDF invariants in the sense of constituting a closed algebra of quantum operators that commute with the constraints, not all of them have a well defined application on all states of the string's Hilbert space. Namely, even though they consist of well-behaved DDF operators, these appear in infinite sums and divergences may occur when acting with these on some states.

Acknowledgments

I am grateful to Robert Graham for his comments on the ideas presented here as well as to Rainald Flume for interesting discussion about Pohlmeyer invariants while these ideas were being developed.

This work was supported by SFB/TR 12.

References

[1] T. Maeda, T. Nakatsu and T. Oonishi, *Non-linear field equation from boundary state formalism*, hep-th/0312260.

[2] K. Murakami and T. Nakatsu, *Open Wilson lines as states of closed string*, *Prog. Theor. Phys.* **110** (2003) 285 [hep-th/0211232].

[3] U. Schreiber, *Nonabelian 2-forms and loop space connections from sCFT deformations*, hep-th/0407122.

[4] K. Pohlmeyer and M. Trunk, *The invariant charges of the Nambu-Goto string theory: quantization of non-additive composition laws*, *Int. J. Mod. Phys.* **A 19** (2004) 115 [hep-th/0206061].

[5] C. Meusburger and K.H. Rehren, *Algebraic quantization of the closed bosonic string*, *Commun. Math. Phys.* **237** (2003) 69 [math-ph/0202041].

[6] K. Pohlmeyer, *The nambu-goto theory of closed bosonic strings moving in 1+3-dimensional minkowski space: the quantum algebra of observables*, *Annalen Phys.* **8** (1999) 19–50 [hep-th/9805057].

[7] K. Pohlmeyer and K.-H. Rehren, *The invariant charges of the Nambu-Goto theory: their geometric origin and their completeness*, *Commun. Math. Phys.* **114** (1988) 177.

[8] T. Thiemann, *The lqg string: loop quantum gravity quantization of string theory, I. Flat target space*, hep-th/0401172.

[9] U. Schreiber, *Ddf and Pohlmeyer invariants of (super)string*, *J. High Energy Phys.* **05** (2004) 027 [hep-th/0403260].

[10] U. Schreiber, *Pohlmeyer invariants are expressible in terms of DDF invariants*, math-ph/0404007.

[11] K. Hashimoto, *Generalized supersymmetric boundary state*, *J. High Energy Phys.* **04** (2000) 023 [hep-th/9909095].

[12] K. Hashimoto, *Corrections to D-brane action and generalized boundary state*, *Phys. Rev.* **D 61** (2000) 106002 [hep-th/9909027].

[13] E.D. Giudice, P.D. Vecchia, and A. Fubini, *General properties of the dual resonance model*, *Ann. Phys. (NY)* **70** (1972) 378.

[14] U. Schreiber, *On deformations of 2D SCFTs*, *J. High Energy Phys.* **06** (2004) 058 [hep-th/0401175].

[15] U. Schreiber, *Loop space methods in string theory*, http://www-stud.uni-essen.de/~sb0264/looptalk.pdf.

[16] O.D. Andreev and A.A. Tseytlin, *Partition function representation for the open superstring effective action: cancellation of mobius infinities and derivative corrections to Born-Infeld lagrangian*, *Nucl. Phys.* **B 311** (1988) 205.

JHEP10(2004)035